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Unsteady current-induced perturbation of a magnetically contained magnetohydrodynamic flow

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A time-dependent current distribution is abruptly switched on, initiating an unsteady disturbance of a two-phase magnetohydrodynamic configuration. This comprises a magnetically permeated, conducting fluid flow contained by a vacuum magnetic field, from within which the source current radiates. An exact solution to the proposed part-time problem is constructed for the magnetic line distortion of the vacuum field.

If the source current is oscillatory, two progressive, non-dissipative waves are normally encountered, these being superposed upon an infinite discrete set of terms obeying separate rules of decay. Propagation occurs longitudinally, parallel to the flow, but with amplitudes dependent on the transverse variable. The waves advance behind two fronts, the faster of which always travels downstream. Depending on whether the flow speed exceeds or is exceeded by (or equals) $\sqrt{2} \times$ the quadratic mean of both Alfvén speeds involved, the slower front proceeds, respectively, downstream or upstream (or disappears, in which case, only one wave exists). Along a characteristic (a front path), appropriate contributions depend solely on the transverse co-ordinate, behaving otherwise like Riemann invariants. Contrary to expectation, the net perturbation is continuous across each characteristic. Various steady modes are ultimately attained after an infinite period. The radiation principle is satisfied.

Both travelling waves are vibrationally sustained, vanishing with the source frequency. In such an event, the infinite series result is summable to a closed form. From this, the general solution, corresponding to an arbitrary space–time source distribution, is deduced. Certain characteristic-associated equivalence laws are then established. An asymptotic approximation is made.

1. Introduction

Steady-state, stationary waves are known to exist along the interface of a magnetically contained magnetohydrodynamic flow. This was first demonstrated by Savage (1967, 1970), who studied the steady motion induced by time-independent sources with specific spatial distributions. The specific spatial distribution of Savage (1967) was a magnetic dipole suspended within the confining magnetic field. That of Savage (1970) was an algebraically decaying pressure exerted along the interface. Superposed upon the stationary waves are additional terms found to be decaying parallel to the flow, but otherwise formally represented by Laplace integrals. Hence, only the far-field asymptotic form of Savage's results is truly explicit.

The present paper is concerned with the unsteady motion produced in a rather related two-phase magnetohydrodynamic configuration. This motion commences from an initial instant at which a time-dependent current distribution is suddenly activated. The equilibrium arrangement considered by Savage is partially bounded in the sense that the flow layer is of finite thickness while its confining magnetic field extends transversely to infinity. The present paper focuses on the fully bounded problem with the confining field terminating at a conducting surface.

Special attention is paid to the case where the source current is strictly oscillatory. Three possible distinct patterns of transmission arise. These correspond to the flow speed $U > V$ $(=\sqrt{2}\times$ quadratic mean of both Alfvén speeds involved), $U < V$, $U = V$. The propagation is virtually one-dimensional, being aligned with the flow and magnetic fields. Amplitudes, however, vary with the transverse distance measured from the interface. In particular, certain 'quasi-Riemann invariants' depend solely on this transverse co-ordinate along their assigned conveying characteristics (cf. Courant & Hilbert 1962). There are no actual Riemann invariants. Furthermore, the solution is continuous across each characteristic. Evidently, there lacks a close identification with the behaviour of one-dimensional Cauchy waves. Nevertheless, non-dissipative wave functions do emerge. One such wave invariably travels downstream behind a fast front which advances with the group velocity $U+V$ along the appropriate characteristic path. Provided $U \neq V$, another conserved wave appears, moving downstream/upstream according as $U \gtrless V$, behind a slow front. The latter progresses with the group velocity $U-V$ along its characteristic path.

There is always a permanently steady component which pulsates at the source frequency. Its complement generally remains unsteady at finite time, but ultimately develops steady forms after an infinite period along a constant velocity path. These forms vary with the path, being either independent of time and the longitudinal co-ordinate, or purely vibratory. In the far-field zones, they eventually cancel out.

If relativistic effects are accounted for, the radiation principle can be explained within the context of the present analysis. A group velocity interpretation of Lighthill (1960, 1965, 1967) is also verified. Lighthill applied the radiation condition to solve steady-state wave problems uniquely. Instead, we adopt his (1960, appendix B) method, which effectively substitutes zero initial values, corresponding to a part-time hypothesis, for the radiation condition. The consistency of our unsteady results with the radiation principle is then a natural outcome of this hypothesis.

The existence of characteristics, in association with time-dependent, flow-aligned propagation is somewhat reminiscent of the one-dimensional wave equation in particular (though appreciable differences do arise, as indicated earlier), and hyperbolic differential equations in general. But the fluid motion and that of its confining field are both elliptic, being, respectively, Laplacian and Poissonian in two space variables. This ellipticity plays a relatively subdued role, but significantly influences amplitudes. The dominant 'pseudo-hyperbolicity' and time dependence are apparently acquired through an interface boundary condition that is, essentially, an integro-differential equation. During a previous steady-state investigation of a magnetically pinched cylindrical jet of ionized gas (1972), the author also noticed the occurrence of spatial, 'pseudo-hyperbolic' modes within elliptic fields. The cause was traced to boundary interaction.

In the equilibrium configuration, the fluid layer has the same depth as that of the confining magnetic field. This special geometry produces non-dispersiveness, and facilitates the derivation of an exact solution. This is particularly important in the present system, because the latter propagates distinct, characteristic-bounded, variable zones; the phenomenon encountered within a zone in the vicinity of the source being quite different from one far away. In particular, steady-state modes differ in the various zones. Furthermore, this variety leads to interesting interpretations and subsequent comparisons of the radiation principle. Most of these factors would have been lost with an asymptotic solution (which is, incidentally, deduced as a corollary in § 9). If the relevant depths are unequal, dispersion prevails. This would introduce new difficulties, and place a major obstacle in the way of an exact evaluation. An asymptotic approximation may be possible, but it would restrict attention to essentially the region about the fastest front. However, it is anticipated that the actual phenomena will be considerably different from those described by the present theory. Such differences will be investigated in a subsequent paper.

2. Equations of motion

We consider a completely uniform state, wherein a streaming layer $-1 \leqslant y < 0$ ($|x| < \infty$) of incompressible, inviscid, perfectly conducting fluid is permeated by a magnetic field $\mathbf{H}_0 = (H_0, 0)$ and fringes, along $y = 0$, upon another magnetic field $\mathbf{B}_0 = (B_0, 0)$, parallel to $\mathbf{H}_0$ and traversing a vacuous space: $0 < y \leqslant 1$ ($|x| < \infty$). Throughout, reference is made to a two-dimensional $\mathbf{r} = (x, y)$ frame and the time t co-ordinate. Both parallel fields are aligned with the flow velocity $\mathbf{U} = (U, 0)$ of the fluid, whose pressure and density are, respectively, p_0, ρ_0, and whose magnetic permeability is μ. The entire configuration is enclosed within two planar boundaries parallel to the interface $y = 0$, viz. a rigid base throughout $y = -1$ for the fluid, and a perfectly conducting wall surface along $y = 1$ covering the vacuum field. For all purposes, we assume $U \geqslant 0$.

Suppose small perturbations are induced into the equilibrium system described by a source of weak current $I(\mathbf{r}, t)$, propagated perpendicular to the x, y plane along an infinite length conductor stationed in the vacuum field. We assume that $I(\mathbf{r}, t)$ is either spatially distributed over a finite subdomain of $0 < y < 1$ ($|x| < \infty$) or that it tends continuously to zero as $|x| \to \infty$ in $0 < y < 1$. Variations induced, upon the uniform state, by this current source are correspondingly weak and therefore satisfy linearized equations. A non-relativistic treatment is proposed. Let p, $\mathbf{u} = (u_x, u_y)$, $\mathbf{H} = (H_x, H_y)$ be the respective variations of the fluid pressure, velocity and immersed magnetic field. Within $-1 \leqslant y < 0$, the governing equations are then

$$\operatorname{div} \mathbf{u} = 0, \quad \operatorname{div} \mathbf{H} = 0, \tag{2.1}$$

$$D\mathbf{H}/Dt = H_0 \, \partial \mathbf{u}/\partial x, \tag{2.2}$$

$$\rho_0 D\mathbf{u}/Dt + \operatorname{grad}(p + \mu H_0 H_x/4\pi) = (\mu H_0/4\pi)\, \partial \mathbf{H}/\partial x, \tag{2.3}$$

where $D/Dt = \partial/\partial t + U\partial/\partial x$. Now, if η is the (transverse) elevation of a magnetic line of force of the total field $\mathbf{H}_0 + \mathbf{H}$ so that

$$H_0 \, \partial\eta/\partial x = H_y, \tag{2.4}$$

then

$$\frac{\partial}{\partial x}\left(\frac{D\eta}{Dt} - u_y\right) = 0. \tag{2.5}$$

The rigid boundary condition on the fluid is

$$u_y = 0 \quad \text{along} \quad y = -1. \tag{2.6}$$

All fluid disturbances originate from its incidence, at the interface $y = 0$, of the radiation field within $0 < y \leqslant 1$. This radiation is governed by

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0, \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 4\pi I(\mathbf{r}, t), \tag{2.7}$$

with

$$B_y = 0 \quad \text{along} \quad y = 1, \tag{2.8}$$

wherein $\mathbf{B} = (B_x, B_y)$ is the magnetic field deviation from $\mathbf{B}_0$. If ξ is the elevation of a magnetic line of force of the total field $\mathbf{B}_0 + \mathbf{B}$, then

$$B_0 \, \partial\xi/\partial x = B_y. \tag{2.9}$$

All displacement current terms normally associated with electromagnetic theory are virtually zero under a non-relativistic hypothesis. In particular, then, (2.7) contains no time derivative. So the quantity $\mathbf{B}$ acquires its time dependence partly through its in-phase variation with the source current $I(\mathbf{r}, t)$, and partly through a reflective effect caused by the fluid which, under exposure to radiation, executes an important unsteady motion inherent from the time derivatives present in (2.2) and (2.3). Part of the fluid motion is also in phase with the source current. The reflective effect is largely due to conditions at the interface. Its deformation (i.e. the flow profile) is, in view of (2.5), $(\eta)_{y=0}$. Consequently, the continuity of the normal component of magnetic field requires that

$$\xi = \eta \quad \text{along} \quad y = 0. \tag{2.10}$$

The condition of pressure balance is reducible to

$$p + \mu H_0 H_x/4\pi = B_0 B_x/4\pi \quad \text{along} \quad y = 0, \tag{2.11}$$

because in the undisturbed equilibrium,

$$p_0 + \mu H_0^2/8\pi = B_0^2/8\pi, \quad \text{i.e.} \quad a_0^2 - a^2 = 2p_0/\rho_0, \tag{2.12}$$

where $a = (\mu H_0^2/4\pi\rho_0)^{\frac{1}{2}}$ is the internal Alfvén speed within the fluid, and $a_0 = (B_0^2/4\pi\rho_0)^{\frac{1}{2}}$ is an Alfvén speed for the interface.

3. A space–time Fourier analysis

Let
$$J(\mathbf{r},t) = -(4\pi/B_0)\, I(\mathbf{r},t).$$

To solve the space–time problem posed, we apply a double Fourier transformation:
$$J^*(\omega,\alpha;y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J(\mathbf{r},t)\exp\{-i(\alpha x+\omega t)\}\,dx\,dt, \tag{3.1}$$

the superscript * indicating the Fourier transform, in this case of $J(\mathbf{r},t)$. Whereupon, (2.7) and (2.9) yield
$$\partial^2\xi^*/\partial y^2 - \alpha^2\xi^* = -J^*(\omega,\alpha;y), \tag{3.2}$$
whose general solution is
$$\xi^* = C(\omega,\alpha)\exp\{\alpha y\} + D(\omega,\alpha)\exp\{-\alpha y\} - \alpha^{-1}\int^{y}\sinh[\alpha(y-Y)]\,J^*(\omega,\alpha;\,Y)\,dY, \tag{3.3}$$

in terms of arbitrary $C(\omega,\alpha)$ and $D(\omega,\alpha)$. These are determinable from (via (2.8) and (2.9))
$$\xi^* = 0 \quad \text{along} \quad y = 1, \tag{3.4}$$

together with a second condition along $y = 0$. The latter must be established from the fluid and interface equations.

From (2.1) and (2.3), we have
$$(\partial^2/\partial y^2 - \alpha^2)(p^* + \mu H_0 H_x^*/4\pi) = 0,$$

while (2.6), together with the y components of (2.2) and (2.3), implies
$$\frac{\partial}{\partial y}(p^* + \mu H_0 H_x^*/4\pi) = 0 \quad \text{along} \quad y = -1.$$

Also, via (2.7), (2.9) and (2.11),
$$p^* + \mu H_0 H_x^*/4\pi + \rho_0 a_0^2\,\partial\xi^*/\partial y = 0 \quad \text{along} \quad y = 0.$$

Consequently, one obtains
$$p^* + \mu H_0 H_x^*/4\pi = -\rho_0 a_0^2(\partial\xi^*/\partial y)_{y=0}\operatorname{sech}\alpha\cosh[\alpha(y+1)]. \tag{3.5}$$

Now, Fourier transforming (2.2)–(2.4) leads to
$$\begin{aligned}[(\omega+U\alpha)^2 - a^2\alpha^2]\,\eta^* &= \rho_0^{-1}\frac{\partial}{\partial y}(p^* + \mu H_0 H_x^*/4\pi)\\ &= -\alpha a_0^2(\partial\xi^*/\partial y)_{y=0}\operatorname{sech}\alpha\sinh[\alpha(y+1)],\end{aligned} \tag{3.6}$$

via (3.5). Whence, by virtue of (2.10),
$$a_0^2\,\alpha\tanh\alpha\,\partial\xi^*/\partial y + [(\omega+U\alpha)^2 - a^2\alpha^2]\,\xi^* = 0 \quad \text{along} \quad y = 0, \tag{3.7}$$

the required lower condition on ξ^*. The manipulation from (3.3), (3.4) and (3.7), to the unique form of ξ^*, is highly complicated. We merely display the final result:

$$\begin{aligned}\xi^*(\omega,\alpha;y) = {} & \frac{\sinh[\alpha(1-y)]}{\alpha\sinh\alpha}\int_0^y \sinh(\alpha Y)\,J^*(\omega,\alpha;\,Y)\,dY \\ & +\frac{\sinh(\alpha y)}{\alpha\sinh\alpha}\int_y^1 \sinh[\alpha(1-Y)]\,J^*(\omega,\alpha;\,Y)\,dY \\ & +\frac{2a_0^2\alpha\sinh[\alpha(1-y)]}{\sinh(2\alpha)\,[V^2\alpha^2-(\omega+U\alpha)^2]}\int_0^1 \sinh[\alpha(1-Y)]\,J^*(\omega,\alpha;\,Y)\,dY, \end{aligned}\tag{3.8}$$

where $V = (a_0^2+a^2)^{\frac{1}{2}}$ (i.e. $\sqrt{2}\times$ the quadratic mean of a_0 and a).

In this paper, let us confine our interest to the part-time perturbation motion, which is effectively stagnant until an instant $t = 0$, i.e. $\xi \equiv 0$ throughout $t < 0$. This is provided for, if the Fourier inversion for ξ is defined by (cf. Lighthill 1960, appendix B; 1967)

$$\xi(\mathbf{r},t) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty+i\gamma}^{\infty+i\gamma} \xi^*(\omega,\alpha;y)\exp\{i(\alpha x+\omega t)\}\,d\omega, \tag{3.9}$$

where for each α in $(-\infty,\infty)$ and for each y in $[0,1]$,

$$\gamma < \mathrm{Im}\,(\text{lowest singularity of } \xi^*(\omega,\alpha;y) \text{ in the } \omega \text{ plane}). \tag{3.10}$$

Since (3.8) reveals that $\xi^*(\omega,\alpha;y)$ normally possesses (at least two) ω singularities, at $\omega = (V-U)\alpha$ and $-(V+U)\alpha$, which are permanently real on $-\infty < \alpha < \infty$, (3.10) automatically implies $\gamma < 0$ in the first place. Naturally, the inducing source current must remain inoperative until $t = 0$, when it is abruptly switched on. As an unstable in-phase contribution to the motion is undesirable, we propose that

$$|J(\mathbf{r},t)| < \infty \quad \text{over} \quad 0 < t \leqslant \infty. \tag{3.11}$$

It follows that $J^*(\omega,\alpha;y)$ is analytic throughout the lower half-plane $\mathrm{Im}\,\omega < 0$. Consequently, by the same reasoning associated with (3.9) and (3.10), plus the necessity that $\gamma < 0$, the inversion rule

$$J(\mathbf{r},t) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty-i\gamma}^{\infty+i\gamma} J^*(\omega,\alpha;y)\exp\{i(\alpha x+\omega t)\}\,d\omega \tag{3.12}$$

(with the same γ as in (3.9)) does satisfy the requirement that $J(\mathbf{r},t) \equiv 0$ during $t < 0$, no matter where the remaining singularities of $\xi^*(\omega,\alpha;y)$ are located. For stability, however, *none* of these singularities must lie inside $\mathrm{Im}\,\omega < 0$, in which case it is necessary and sufficient that $\gamma < 0$. The part-time system proposed is comparable to a Cauchy-type radiation problem complying with zero initial conditions (Courant & Hilbert 1962).

The inversion (cf. (3.9)) for $\eta(\mathbf{r},t)$ in the fluid region $-1 \leqslant y \leqslant 0$ can be accomplished through (3.6) and (3.8). However, as there is no reason to expect any staggering departure from the analysis for $\xi(\mathbf{r},t)$, we shall, as from now, merely concentrate on the latter.

4. Induction by an oscillating current

Consider a source current which is oscillatory with time from the instant it is activated, i.e.

$$J(\mathbf{r},t) = J(\mathbf{r})\exp\{i\lambda t\}\,H(t), \tag{4.1}$$

λ being a real constant frequency, and $H(t)$ being the Heaviside unit step function. It then follows from (3.12) (via an ω contour integration, say) that the Fourier transform

$$J^*(\omega,\alpha;y) = \frac{\int_{-\infty}^{\infty} J(\mathbf{r})\exp\{-i\alpha x\}\,dx}{4\pi^2 i(\omega-\lambda)}. \tag{4.2}$$

Applying this to (3.8) and inverting ξ^* by (3.9) leads to

$$\xi(\mathbf{r},t) = \int_{-\infty}^{\infty} dX\int_0^y J(\mathbf{R})\,K(\mathbf{r},t|\mathbf{R})\,dY + \int_{-\infty}^{\infty} dX\int_y^1 J(\mathbf{R})\,\hat{K}(\mathbf{r},t|\mathbf{R})\,dY$$
$$+\int_{-\infty}^{\infty} dX\int_0^1 J(\mathbf{R})\,L(\mathbf{r},t|\mathbf{R})\,dY \quad (0 \leqslant y \leqslant 1), \tag{4.3}$$

where the vector $\mathbf{R} = (X,\, Y)$ with $0 < Y < 1$;

$$K(\mathbf{r},t|\mathbf{R}) = K(x,y,t|X,\,Y): \quad \text{for} \quad Y \leqslant y,$$

$$K(\mathbf{r},t|\mathbf{R}) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty} \frac{\sinh[\alpha(1-y)]\sinh(\alpha Y)}{\alpha\sinh\alpha}$$
$$\times\exp\{i\alpha(x-X)\}\,d\alpha\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{\exp\{i\omega t\}}{\omega-\lambda}\,d\omega; \tag{4.4}$$

$$\hat{K}(\mathbf{r},t|\mathbf{R}) = K(x,\,Y,t|X,y) \quad \text{for} \quad y \leqslant Y, \tag{4.5}$$

derivable from (4.4) by interchanging y and Y; and

$$L(\mathbf{r},t|\mathbf{R}) = \frac{a_0^2}{2\pi^2 i}\int_{-\infty}^{\infty} \frac{\alpha\sinh[\alpha(1-y)]\sinh[\alpha(1-Y)]}{\sinh(2\alpha)}\exp\{i\alpha(x-X)\}\,d\alpha$$
$$\times\int_{-\infty+i\gamma}^{\infty+i\gamma} \frac{\exp\{i\omega t\}\,d\omega}{(\omega-\lambda)\,[V^2\alpha^2-(\omega+U\alpha)^2]}. \tag{4.6}$$

We note, in passing, that if the oscillatory source current is conducted along a thin wire (forming a current filament) passing through the point $\mathbf{R}$ in the vacuum field, i.e.

$$J(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{R}) \equiv \delta(x-X)\,\delta(y-Y) \quad (0 < Y < 1), \tag{4.7}$$

with δ denoting the Dirac delta function, then

$$\xi(\mathbf{r},t) = K(\mathbf{r},t|\mathbf{R})+L(\mathbf{r},t|\mathbf{R}) \quad \text{in} \quad y \geqslant Y, \tag{4.8}$$

but

$$\xi(\mathbf{r},t) = \hat{K}(\mathbf{r},t|\mathbf{R})+L(\mathbf{r},t|\mathbf{R}) \quad \text{in} \quad y \leqslant Y. \tag{4.9}$$

Evidently, K, $\hat{K}$ and L are element kernels of a Green's function associated with the pulsating source.

Consider both (inner) ω integrals encountered in (4.4) and (4.6), the former being independent of the outer integration variable α, while the latter is a function of α. Each integrand factor accompanying $\exp\{i\omega t\}$ is a meromorphic (precisely, a rational) function of ω which clearly satisfies a uniform convergence condition of Jordan's lemma in the complex ω plane. Concerning the ω integrand appearing in (4.6), its only singularities are three real, simple poles at $\omega = \lambda$, $(V-U)\alpha$, $-(V+U)\alpha$, all of which lie above the prescribed integral path $(-\infty+i\gamma, \infty+i\gamma)$ since $\gamma < 0$. According as $t > 0$ (or < 0), this path may be completed by an infinite semicircle drawn into the upper half-plane $\operatorname{Im}\omega > \gamma$ (or lower half-plane $\operatorname{Im}\omega < \gamma$), to form a positively (or negatively) described closed contour which obviously encloses all three (or none) of the ω poles. The relevant extended integration, performed along the appropriate choice of semicircle, vanishes by virtue of Jordan's lemma. Whence, in accordance with residue theory, the ω integral of (4.6) possesses the value

$$2\pi i H(t)\left\{\frac{\exp\{i\lambda t\}}{V^2\alpha^2-(\lambda+U\alpha)^2}-\frac{\exp\{i(V-U)\alpha t\}}{2V\alpha[(V-U)\alpha-\lambda]}-\frac{\exp\{-i(V+U)\alpha t\}}{2V\alpha[(V+U)\alpha+\lambda]}\right\}. \tag{4.10}$$

By similar arguments, the value of the ω integral of (4.4) is

$$2\pi i H(t)\exp\{i\lambda t\}. \tag{4.11}$$

The occurrence of the Heaviside factor $H(t)$ confirms the desired part-time effect: $\xi \equiv 0$ throughout $t < 0$. Note that the strictly harmonic forms of (4.10) and (4.11) are consistent with a stable vibrating system.

Substituting (4.10) into (4.6), and introducing the vectors

$$\mathbf{r}_1 = (x_1, y) \equiv (x-(U+V)t, y), \quad \mathbf{r}_2 = (x_2, y) \equiv (x-(U-V)t, y), \tag{4.12}$$

we arrive at

$$L(\mathbf{r},t|\mathbf{R}) = (a_0^2/V)H(t)[\exp\{i\lambda t\}F(V;\mathbf{r}|\mathbf{R}) \\ -F(V;\mathbf{r}_1|\mathbf{R})-\exp\{i\lambda t\}F(-V;\mathbf{r}|\mathbf{R})+F(-V;\mathbf{r}_2|\mathbf{R})], \tag{4.13}$$

where the function $F(V;\mathbf{r}|\mathbf{R}) \equiv F(V;x,y|X,Y)$ is first expressed by the principal value representation of (A 1) and eventually evaluated as an infinite series (A 14). Likewise, $K(\mathbf{r},t|\mathbf{R})$, initially represented by the integral of (A 19), is finally determined by either the infinite series (A 20) or the closed form (A 21). All analytical results and their interpretations are provided in § 5.

5. Characteristic-bounded propagation

If y and Y are interchanged in (4.4) (as well as (A 19)), we obtain an infinite integral convergent in $y \leqslant Y$ ($\mathbf{r} \neq \mathbf{R}$) and representing the kernel $\hat{K}(\mathbf{r},t|\mathbf{R})$ defined by (4.5). Nevertheless, regarding the final form (A 21), established under the assumption $y \geqslant Y$, the variables y and Y (as well as x and X) are interchangeable. Thus we see that

$$\hat{K}(\mathbf{r},t|\mathbf{R}) \equiv K(\mathbf{r},t|\mathbf{R}) \equiv K(\mathbf{R},t|\mathbf{r}), \tag{5.1}$$

with

$$K(\mathbf{r},t|\mathbf{R}) = H(t)(4\pi)^{-1}\exp\{i\lambda t\}\ln\left\{\frac{\sinh^2[\frac{1}{2}\pi(x-X)]+\sin^2[\frac{1}{2}\pi(y+Y)]}{\sinh^2[\frac{1}{2}\pi(x-X)]+\sin^2[\frac{1}{2}\pi(y-Y)]}\right\}. \tag{5.2}$$

Therefore, corresponding to the pulsatory source function (4.1), the ξ solution is representable, in the infinite strip Ξ: $\{0 \leqslant y \leqslant 1,\ -\infty < x < \infty\}$, by

$$\xi(\mathbf{r},t) = \iint_{\Xi} J(\mathbf{R})\,[K(\mathbf{r},t|\mathbf{R}) + L(\mathbf{r},t|\mathbf{R})]\,d\mathbf{R}, \tag{5.3}$$

(cf. (4.3)), $d\mathbf{R}$ being an area element.

The complementary kernel $L(\mathbf{r},t|\mathbf{R})$ is determined for $U \neq V$ by applying (A 14) to (4.13). First, let us define

$$L_1(\mathbf{r},t|\mathbf{R}) = W_1 H(t)\,H(x-X)\,H(X-x_1), \tag{5.4}$$

$$L_2(\mathbf{r},t|\mathbf{R}) = \begin{cases} -W_2 H(t)\,H(x-X)\,H(X-x_2) & \text{if} \quad U > V, \quad (5.5)\\ W_2 H(t)\,H(X-x)\,H(x_2-X) & \text{if} \quad U < V, \quad (5.6)\end{cases}$$

where x_1, x_2 are defined by (4.12), and

$$W_1 \equiv W_1(x_1-X, y, Y) = \frac{ia_0^2 \sinh[\alpha_1(1-y)]\sinh[\alpha_1(1-Y)]\exp\{i\alpha_1(x_1-X)\}}{V(U+V)\sinh(2\alpha_1)}, \tag{5.7}$$

$$W_2 \equiv W_2(x_2-X, y, Y) = \frac{ia_0^2 \sinh[\alpha_2(1-y)]\sinh[\alpha_2(1-Y)]\exp\{i\alpha_2(x_2-X)\}}{V(U-V)\sinh(2\alpha_2)}, \tag{5.8}$$

with $\alpha_1 = -\lambda/(U+V)$, $\alpha_2 = -\lambda/(U-V)$. Furthermore, we introduce

$$\begin{Bmatrix}\xi_\nu^+\\ \xi_\nu^-\end{Bmatrix} = iH(t)\frac{a_0^2}{V}\sum_{n=1}^{\infty}(-1)^n \sin[\tfrac{1}{2}n\pi(1-y)]\sin[\tfrac{1}{2}n\pi(1-Y)]\begin{Bmatrix}f_{\nu n}^+\\ f_{\nu n}^-\end{Bmatrix} \quad (\nu = 1,2,3), \tag{5.9}$$

where

$$f_{1n}^{\pm} \equiv f_{1n}^{\pm}(x_1-X) = \frac{\exp\{-\frac{1}{2}n\pi|x_1-X|\}}{in\pi(U+V)\pm 2\lambda}, \tag{5.10}$$

$$f_{2n}^{\pm} \equiv f_{2n}^{\pm}(x_2-X) = -\frac{\exp\{-\frac{1}{2}n\pi|x_2-X|\}}{in\pi(U-V)\pm 2\lambda}, \tag{5.11}$$

$$f_{3n}^{\pm} \equiv f_{3n}^{\pm}(x-X,t) = \exp\{i\lambda t\}\exp\{-\tfrac{1}{2}n\pi|x-X|\} \times\left[\frac{1}{in\pi(U-V)\pm 2\lambda} - \frac{1}{in\pi(U+V)\pm 2\lambda}\right]. \tag{5.12}$$

Thus, in terms of (5.4)–(5.12) and for $U \neq V$

$$L(\mathbf{r},t|\mathbf{R}) = L_1(\mathbf{r},t|\mathbf{R}) + L_2(\mathbf{r},t|\mathbf{R}) + \xi_1^+ H(x_1-X) + \xi_1^- H(X-x_1) + \xi_2^+ H(x_2-X) + \xi_2^- H(X-x_2) + \xi_3^+ H(x-X) + \xi_3^- H(X-x). \tag{5.13}$$

The quantity W_ν ($\nu = 1, 2$) expressed by (5.7), (5.8) represents a non-dissipative travelling wave of wavelength $2\pi/|\alpha_\nu|$. Its passage is restricted to the x direction, but with a variable y-dependent amplitude. Starting from the interface $y = 0$, this amplitude decreases with increasing y, till it vanishes along the conducting wall surface $y = 1$. Both W_1 and W_2 are associated with a non-dispersive, one-dimensional wave system: relative to the positive x direction,

$$\text{phase velocity} = \text{group velocity} = U+V,\ U-V,$$

respectively. The terms ξ_ν^+, ξ_ν^- (ν = 1, 2, 3) are each an infinite sum of sinusoidal functions of y. They are dissipative, but only in the sense that

$$\begin{gathered}\xi_\nu^+ \to 0, \quad \xi_\nu^- \to 0 \quad \text{as} \quad |x_\nu - X| \to \infty \quad (\nu = 1,2),\\ \xi_3^+ \to 0, \quad \xi_3^- \to 0 \quad \text{as} \quad |x-X| \to \infty.\end{gathered} \tag{5.14}$$

Like W_ν, ξ_ν^+ and ξ_ν^- ($\nu = 1, 2$) are longitudinally propagative, but unlike W_ν, they decay as $t \to \infty$, except near one of two distinct straight lines in the x, t ($t > 0$) plane, viz. the γ_1 characteristic

$$X = x_1 \; (\equiv x - (U+V)t),$$

or the γ_2 characteristic

$$X = x_2 \; (\equiv x - (U-V)t).$$

On the other hand, ξ_3^+ and ξ_3^- are fully stationary and steadily oscillatory in phase, with the source current at frequency λ. *Along the γ_1 characteristic, the quantities W_1, ξ_1^+, ξ_1^- are propagated independently of x and t*, from an origin at $(x, t) = (X, 0)$ and *with the fast group velocity $U+V$*. Likewise, along the γ_2 characteristic, W_2, ξ_2^+, ξ_2^- are also propagated independently of x and t from the same origin, but with the *slow group velocity $U-V$*. There is, clearly, an analogy with the conveying of Riemann invariants along the characteristics of hyperbolic differential equations (Courant & Hilbert 1962). In the present situation, however, our '*quasi-Riemann invariants*' are all y dependent. The origin $(x, t) = (X, 0)$ corresponds to the physical source plane containing the current filament through $\mathbf{R}$ at its instant ($t = 0$) of activation (cf. (4.1), (4.7)). Thus, based on an x, t diagram, all perturbations virtually commence at this instantaneous source plane. The kernel $L(\mathbf{r}, t|\mathbf{R})$ can be represented throughout time $t > 0$ (avoiding the initial instant $t = 0$) on the x, t diagram.

First, suppose $U > V$. Both γ_1 and γ_2 characteristics are downstream (i.e. $x > X$) inclined. There are four distinct regions of separation wherein (via (5.4), (5.5), (5.13))

$$L(\mathbf{r}, t|\mathbf{R}) = \begin{cases} [0] + (\xi_1^- + \xi_2^- + \xi_3^-) \quad \text{in} \quad X > x \quad \text{(upstream)}, & (5.15) \\ [W_1 - W_2] + (\xi_3^+ + \xi_1^- + \xi_2^-) \quad \text{in} \quad x > X > x_2 \\ \qquad\qquad \text{(near downstream)}, & (5.16) \\ [W_1] + (\xi_2^+ + \xi_3^+ + \xi_1^-) \quad \text{in} \quad x_2 > X > x_1 \\ \qquad\qquad \text{(mid downstream)}, & (5.17) \\ [0] + (\xi_1^+ + \xi_2^+ + \xi_3^+) \quad \text{in} \quad x_1 > X \\ \qquad\qquad \text{(far downstream)}. & (5.18) \end{cases}$$

The x, t pattern is depicted in figure 1. The [ ] quantities of (5.16) and (5.17) denote permanently undamped, progressive waves. The propagation of W_1 is spaced over a downstream expanding interval $x > X > x_1$, behind a *fast front* $X = x_1$ advancing with the group velocity $U+V$ (i.e. along the γ_1 characteristic). Behind a *slow front* $X = x_2$, which progresses downstream (along the γ_2 characteristic) with the group velocity $U - V$ (> 0), $-W_2$ is superposed upon W_1 within the expanding near-downstream subinterval $x > X > x_2$ (cf. (5.16)). Neither W_1 nor W_2 ever gets transported upstream, as implied by [ ] $\equiv$ [0] in $X > x$ (cf. (5.15)). Likewise, they have not yet penetrated into the far-downstream region $x_1 > X$ beyond the fast front. Superposed upon the entire field of propagation are the ( ) quantities, each composed of terms which obey the dissipation laws of (5.14). Thus, for example, when $t \gg 0$

$$L(\mathbf{r}, t|\mathbf{R}) \sim \begin{cases} \xi_1^+ \quad \text{near} \quad x_1 = X + 0_+, & (5.19) \\ W_1 + \xi_1^- \quad \text{near} \quad x_1 = X + 0_-, & (5.20) \end{cases}$$

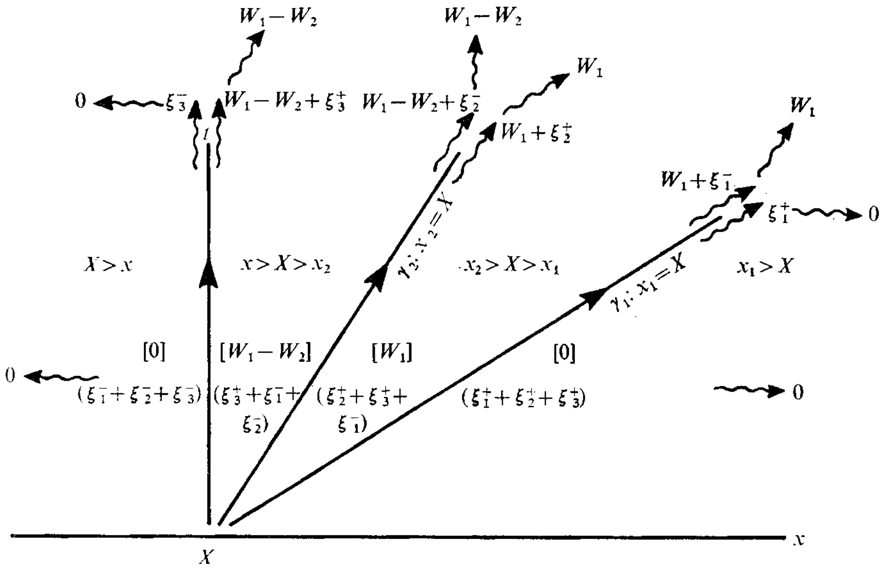

FIGURE 1. The x, t pattern for the propagation of $L(\mathbf{r}, t|\mathbf{R})$ during $t > 0$ in the case $U > V$. The longitudinally transported waves W_1 and W_2 are confined respectively to a downstream zone $x > X > x_1$ behind the γ_1 characteristic $x_1 = X$, and to a near-downstream zone $x > X > x_2$ behind the γ_2 characteristic $x_2 = X$. The values associated with the wriggly arrows correspond to the appropriate asymptotic behaviour for large $|x - X|$, and for large t close about, as well as diverging from, the t axis, the γ_1 and γ_2 characteristics.

$$L(\mathbf{r}, t|\mathbf{R}) \sim \begin{cases} W_1 + \xi_2^+ & \text{near} \quad x_2 = X + 0_+, \qquad (5.21) \\ W_1 - W_2 + \xi_2^- & \text{near} \quad x_2 = X + 0_-, \qquad (5.22) \end{cases}$$

$$L(\mathbf{r}, t|\mathbf{R}) \sim \begin{cases} W_1 - W_2 + \xi_3^+ & \text{near} \quad x = X + 0_+, \qquad (5.23) \\ \xi_3^- & \text{near} \quad x = X + 0_-. \qquad (5.24) \end{cases}$$

This behaviour is indicated in figure 1 by the far pairs of nearly parallel arrows (wriggly, for example, to represent travelling observations that are alternatively retarding and accelerating), described about the γ_1, γ_2 characteristics and the t axis. They record the possible asymptotic developments measured, at large t, by three pairs of L observers, moving independently since the instant $t = 0$, with small variations of velocities about $U + V$, $U - V$, 0 respectively. Suppose the observer who is experiencing the effect (5.20), slightly behind the fast front $x_1 = X$, reduces his mean velocity from approximately $U + V$ to about $U - V$, the process being represented by an arrow wriggling, nearly parallel to the γ_2 characteristic, from the reading $W_1 + \xi_1^-$ towards the reading W_1. Correspondingly, suppose the observer who is exposed to the effect (5.21), slightly ahead of the slow front $x_2 = X$, accelerates to a mean velocity about $U + V$. Evidently these observers, both inside the mid-downstream zone, eventually meet, at which point, they register the same effect, roughly W_1. A similar significance can be attached to the two large t readings $W_1 - W_2 + \xi_3^+$ and $W_1 - W_2 + \xi_2^-$ deviating

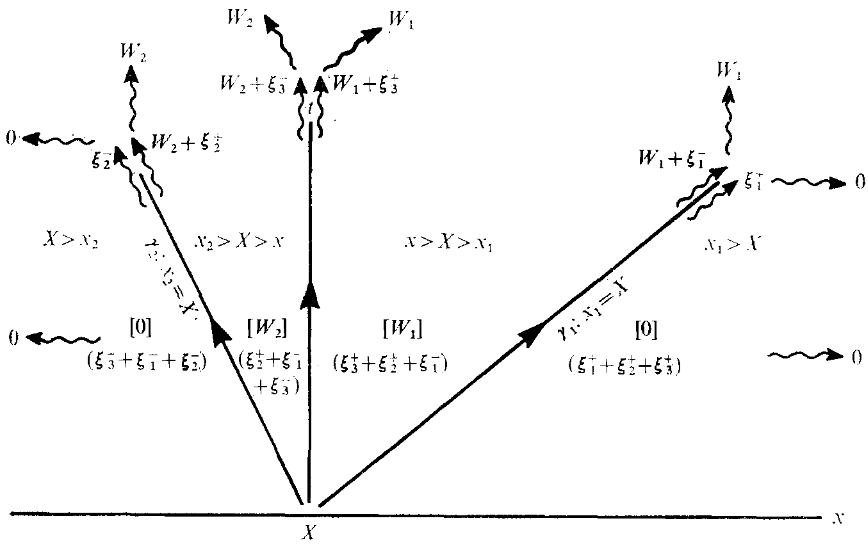

FIGURE 2. The x, t pattern for $U < V$. The γ_1 characteristic remains inclined downstream, while the γ_2 characteristic becomes upstream inclined. The waves W_1 and W_2 are observed within, respectively, the near-downstream zone $x > X > x_1$ and the near-upstream zone $x_2 > X > x$.

towards $W_1 - W_2$ from $x \simeq X + 0_+$ and $x_2 \simeq X + 0_-$ within the near-downstream zone. Also, as denoted by the nearly horizontal wriggly arrows,

$$L(\mathbf{r}, t|\mathbf{R}) \to 0 \quad \text{uniformly as} \quad |x - X| \to \infty \tag{5.25}$$

throughout either the upstream zone $x < X$ or the far-downstream zone $x_1 > X$, corresponding to the extreme far-field developments.

Next, suppose $U < V$. Then the γ_1 characteristic inclines downstream, while the γ_2 characteristic inclines upstream. From (5.4), (5.6) and (5.13), we obtain

$$L(\mathbf{r}, t|\mathbf{R}) = \begin{cases} [0] + (\xi_3^- + \xi_1^- + \xi_2^-) & \text{in} \quad X > x_2 \quad \text{(far upstream)}, & (5.26) \\ [W_2] + (\xi_2^+ + \xi_1^- + \xi_3^-) & \text{in} \quad x_2 > X > x \quad \text{(near upstream)}, & (5.27) \\ [W_1] + (\xi_3^+ + \xi_2^+ + \xi_1^-) & \text{in} \quad x > X > x_1 \quad \text{(near downstream)}, & (5.28) \\ [0] + (\xi_1^+ + \xi_2^+ + \xi_3^+) & \text{in} \quad x_1 > X \quad \text{(far downstream)}. & (5.29) \end{cases}$$

Interpretations similar to those for $U > V$ can be provided. Corresponding effects, relative to the x, t plane are explicitly illustrated in figure 2. In particular, W_1 (W_2) is a strictly downstream (upstream) progressive wave, confined to a right-(left)-expanding interval behind the advancing fast front $x_1 = X$ (slow front $x_2 = X$). The previous forms of (5.15) and (5.18) are, at present, preserved, respectively, far upstream and far downstream (cf. (5.26), (5.29)). In both these zones, the rule (5.25) applies. Furthermore, the mid-downstream solution of (5.17) now emerges near-downstream (cf. (5.28)).

When $U = V$, several of the results after (5.3) fail. In this case, we need to employ (A 18) in dealing with the functions $F(-V; \mathbf{r}|\mathbf{R})$ and $F(-V; \mathbf{r}_2|\mathbf{R})$ of (4.13). Thus, if

$$W = (W_1)_{U=V}, \quad \zeta_1^+ = (\xi_1^+)_{U=V}, \quad \zeta_1^- = (\xi_1^-)_{U=V}, \tag{5.30}$$

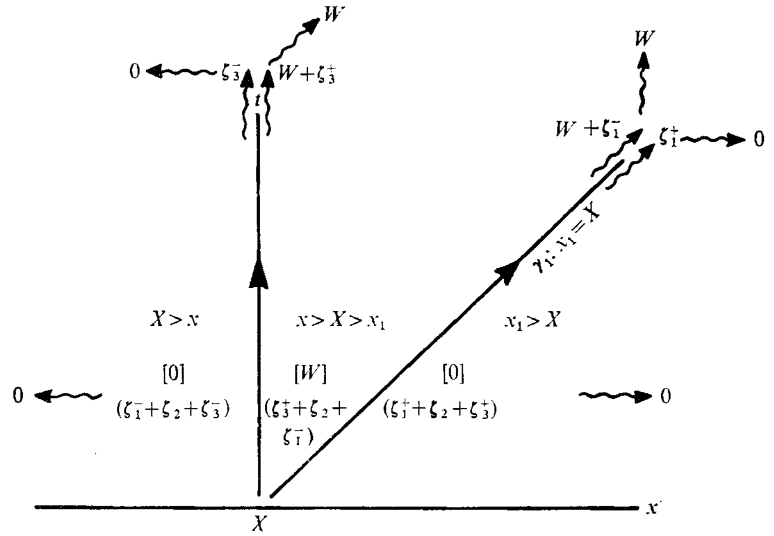

FIGURE 3. The case $U = V$. There is only one characteristic, viz. the γ_1 characteristic, behind which the sole x-directed wave W progresses downstream within $x > X > x_1$.

i.e. the respective values, taken at $U = V$, of the functions defined by (5.7), (5.9) and (5.10),

$$\zeta_3^{\pm} = \frac{iH(t)\,a_0^2 \exp\{i\lambda t\}}{2V} \sum_{n=1}^{\infty} (-1)^{n+1} \sin[\tfrac{1}{2}n\pi(1-y)] \sin[\tfrac{1}{2}n\pi(1-Y)] \times \frac{\exp\{-\tfrac{1}{2}n\pi|x-X|\}}{in\pi V \pm \lambda}; \tag{5.31}$$

$$\zeta_2 = \frac{iH(t)\,a_0^2}{16\lambda V} \frac{(1-\exp\{i\lambda t\}) \sinh[\tfrac{1}{2}\pi(x-X)] \cos(\tfrac{1}{2}\pi y) \cos(\tfrac{1}{2}\pi Y)}{\{\sinh^2[\tfrac{1}{4}\pi(x-X)] + \sin^2[\tfrac{1}{4}\pi(y+Y)]\} \times \{\sinh^2[\tfrac{1}{4}\pi(x-X)] + \cos^2[\tfrac{1}{4}\pi(y-Y)]\}}; \tag{5.32}$$

then whenever $U = V$

$$L(\mathbf{r}, t|\mathbf{R}) = \begin{cases} [0] + (\zeta_1^- + \zeta_2 + \zeta_3^-) & \text{in} \quad X > x \quad \text{(upstream)}, & (5.33) \\ [W] + (\zeta_3^+ + \zeta_2 + \zeta_1^-) & \text{in} \quad x > X > x_1 \quad \text{(near downstream)}, & (5.34) \\ [0] + (\zeta_1^+ + \zeta_2 + \zeta_3^+) & \text{in} \quad x_1 > X \quad \text{(far downstream)}. & (5.35) \end{cases}$$

The only longitudinally travelling wave present is W. It has the wavenumber $\alpha_1 = -\lambda/2U$, and occurs within the near-downstream zone behind the only front $X = x_1$ ($\equiv x - 2Ut$) which is advancing along the downstream-inclined γ_1 characteristic. The functions $\zeta_1^{\pm}$, $\zeta_3^{\pm}$ follow the same laws of decay, viz. (5.14), as $\xi_1^{\pm}$, $\xi_3^{\pm}$; $\zeta_3^{\pm}$ is steadily oscillatory with source frequency λ. The function ζ_2, which contributes throughout $|x| < \infty$, has a steadily oscillatory part superposed upon a time-independent part. If $\mathbf{R}$ is any interior point of the infinite strip Ξ, ζ_2 is an analytic function of x and y over Ξ. In particular,

$$\zeta_2 \to 0 \quad \text{as} \quad |x-X| \to 0 \quad \text{or as} \quad |x-X| \to \infty. \tag{5.36}$$

Figure 3 provides the relevant x, t scheme.

of the ξ values along the symmetric characteristics $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$ at the earlier instant $t = \frac{1}{2}\tau + T$.

In view of (8.3), (8.5) and (8.6) reduce, when $X_0 = X$, to

$$(Q_+)_{\gamma_1(X,T)} = G[2V(t-T), y; Y] = (Q_+)_{\gamma_2(X,T)} \quad (U \geqslant 0). \tag{8.9}$$

This result can be extended and generalized by the following theorem.

THEOREM 2. (An equivalence law of simultaneous propagation.) Suppose the spatial distribution of the impulsive source (8.1) is symmetric about some vertical line $x = X_0$, i.e.

$$J(X_0 - X, Y) = J(X_0 + X, Y). \tag{I}$$

Then along $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$ diverging from the initial point on the given line, the ξ values at any instant are (simultaneously) equal for $U \geqslant 0$:

$$(\xi)_{\gamma_1(X_0,T)} = (\xi)_{\gamma_2(X_0,T)} = \iint_\Xi J(\mathbf{R})\, G[X - X_0 + 2V(t-T), y; Y]\, d\mathbf{R}, \tag{II}$$

with $G(x,y;Y)$ defined by (7.8). Furthermore, if $U = 0$, then along the two symmetric γ_1 and γ_2 characteristics crossing each other on the same line of symmetry $x = X_0$ at any other point, the ξ values are also equal at any instant. (Note that both (8.9) and (II) have obvious applications to (8.8) when $X_0 = X$ and theorem 1 holds respectively.)

Proof. Owing to the second identity of (8.3), (I) implies

$$\iint_\Xi J(\mathbf{R})\, G(X_0 - X, y; Y)\, d\mathbf{R} = 0, \tag{8.10}$$

$$\iint_\Xi J(\mathbf{R})\, G[X_0 - X + 2V(t-T), y; Y]\, d\mathbf{R}$$

$$= \iint_\Xi J(\mathbf{R})\, G[X - X_0 + 2V(t-T), y; Y]\, d\mathbf{R} \tag{8.11}$$

$$= -\iint_\Xi J(\mathbf{R})\, G[X_0 - X - 2V(t-T), y; Y]\, d\mathbf{R}. \tag{8.12}$$

Application to (8.2), (8.5) and (8.6) leads directly to (II). If $U = 0$, and (X_0, T_0) is any point (other than (X_0, T)) on $x = X_0$, then, from (7.7) and (8.4),

$$(Q_+)_{\gamma_1(X_0,T_0)} = G[X_0 - X + 2Vt - V(T_0+T), y; Y] - G[X_0 - X - V(T_0 - T), y; Y],$$

$$(Q_+)_{\gamma_2(X_0,T_0)} = G[X_0 - X + V(T_0 - T), y; Y] - G[X_0 - X - 2Vt + V(T_0+T), y; Y].$$

Following through with an argument like that based upon (8.11) and (8.12), we obtain

$$(\xi)_{\gamma_1(X_0,T_0)} = (\xi)_{\gamma_2(X_0,T_0)}, \tag{8.13}$$

completing the proof. (Note that the identity here does not involve the same integral convolution as (II).)

Suppose the γ_1 and γ_2 characteristics through an arbitrary point (x,t) are $\gamma_1(X_1, T_1)$, $\gamma_2(X_2, T_2)$. Let the γ_2 characteristic through (X_1, T_1) be $\gamma_2(X_3, T_3)$, (X_3, T_3) being the point such that $\gamma_1(X_3, T_3)$ passes exactly through (X_2, T_2). The two pairs of γ_1 and γ_2 characteristics, thus constructed, form what we shall call a normally expected of a fully stable motion generated by a steady force or source. The steady modes encountered are either x, t independent or strictly vibratory. Similar illustrations can be drawn for the cases $U \leqslant V$. Note particularly that, for all $U \geqslant 0$, the steady state ultimately attained by the L kernel in either the extreme downstream or extreme upstream zone along a γ path of the type (6.2) is one of 'silence': $L(\mathbf{r}, \infty|\mathbf{R}) \equiv 0$.

Within the far-downstream/upstream (far-upstream if $U < V$) zone extending to $x = \pm\infty$, the L solution always takes the form $\xi_1^{\pm} + \xi_2^{\pm} + \xi_3^{\pm}$ ($\zeta_1^{\pm} + \zeta_2 + \zeta_3^{\pm}$ if $U = V$), which $\not\equiv 0$. Superposed upon this is $K(\mathbf{r}, t|\mathbf{R})$, also $\not\equiv 0$. In the far field, these non-trivial L and K forms operate the instant the source current (6.1) is switched on at $t = 0$. The immediate impression conveyed is that the Sommerfeld radiation principle is violated. From this principle, it should instead follow that all perturbations (which must originate at the solitary current source) cannot at finite t participate arbitrarily far away, but must be contained within a finite domain expanding about the source plane. The controversy arises because our original non-relativistic formulation of § 2 assumes the inverse of the speed of light to be negligible. Consequently, electromagnetic (or optical) effects, travelling at speeds comparable to that of light and contributing to the motion, are infinitely faster than both fast and slow fronts. The correct interpretation is that $\xi_1^{\mp} + \xi_2^{\mp} + \xi_3^{\mp}$ and $K(\mathbf{r}, t|\mathbf{R})$ represent such infinitely fast electromagnetic effects. This should then explain their immediate appearances, since $t = 0$, at far distances. A reliable test for the radiation principle is through (5.2) and (5.25). They indicate that, far downstream and upstream (far upstream if $U < V$),

$$K(\mathbf{r}, t|\mathbf{R}) + L(\mathbf{r}, t|\mathbf{R}) \to 0 \quad \text{uniformly as} \quad |x - X| \to \infty.$$

This eliminates the possibility of any disturbance being admitted from both infinite ends. Moreover, as $\mathbf{r} \to \mathbf{R}$, $|K(\mathbf{r}, t|\mathbf{R})| \to \infty$ like $|\ln|\mathbf{r} - \mathbf{R}|^2|$, verifying that $K(\mathbf{r}, t|\mathbf{R})$ certainly originates at the filament source (6.1). (In fact, the logarithmic singularity, here, is reminiscent of the unbounded electrostatic field near a steady line current.) All these factors strongly suggest a consistency with the radiation principle. Lighthill (1960, 1965, 1967) applied the radiation principle, in place of initial conditions, to investigate steady wave problems during time $t = \infty$, and arrived at a group velocity interpretation. Consider the eventual steady states of the filament source (6.1) attained by ξ far off the source plane as well as the fast and slow fronts, e.g. about the path (6.2). We have (i) a W_1 wave travelling downstream behind $x_1 = X$ with the positive group velocity $U + V$; (ii) a W_2 wave travelling downstream/upstream behind $x_2 = X$ if, and only if, its group velocity $U - V \gtrless 0$. Note that when the group velocity $U - V = 0$, the W_2 wave disappears. Both phenomena (i) and (ii) agree with Lighthill's conclusion that the group velocity, of an anisotropic wave system, has a positive (outward) component along the position vector, i.e. wave energy is transmitted away from the source. Furthermore, (iii) well beyond the fast front $x_1 = X$ (as well as the slow front $x_2 = X$ in the case $U < V$), a steady zone of 'silence' develops: $\xi \equiv 0$. This also happens upstream when $U \geqslant V$. Consequently, during a steady state, the radiation principle is satisfied in the sense of both Lighthill and Sommerfeld.

As indicated earlier, there appears to be a close resemblance between the propagation pattern and one that is governed by a hyperbolic differential equation in x, t co-ordinates. Now the main equations governing the fluid interior and the vacuum field are (cf. (2.1), (2.3), (2.7)) Laplace's and Poisson's, respectively, both of which are elliptic and time-independent. Our time variation evidently derives from the interface condition (3.7), whose Fourier inverse corresponds to an integro-differential equation. This equation imparts a 'pseudo-hyperbolicity' that is seen to prevail.

7. The general solution

In the general situation, the space–time distribution $J(\mathbf{z}) \equiv J(\mathbf{r},t)\,(\equiv 0$ for $t<0)$ of the suddenly applied source current satisfies (3.11), but otherwise remains *arbitrary*. Throughout, $\mathbf{z} = (\mathbf{r},t) \equiv (x,y,t)$ denotes a three-dimensional (hyperspace) position vector defined on $0 \leqslant y \leqslant 1$. Let $\mathbf{Z} = (\mathbf{R},T) \equiv (X,Y,T)$ with $0 < Y < 1$ and $T \geqslant 0$. The induced part-time ξ perturbation is then expressible, via (3.1), (3.8) and (3.9), as

$$\xi(\mathbf{z}) = \int_0^\infty dT \int_{-\infty}^\infty dX \left[\int_0^y J(\mathbf{Z})\,P(\mathbf{z}|\mathbf{Z})\,dY + \int_y^1 J(\mathbf{Z})\,P(x,\,Y,t|X,y,T)\,dY + \int_0^1 J(\mathbf{Z})\,Q(\mathbf{z}|\mathbf{Z})\,dY\right], \tag{7.1}$$

where $P(\mathbf{z}|\mathbf{Z}) \equiv P(x,y,t|X,\,Y,T)$ and $Q(\mathbf{z}|\mathbf{Z})$ possess the respective forms (4.4), (4.6), but with the common integrand factor $\exp\{i\omega t\}/i(\omega-\lambda)$ replaced by $\exp\{i\omega(t-T)\}$; i.e. $P(\mathbf{z}|\mathbf{Z})$ is derived from $K(\mathbf{r},t|\mathbf{R})$, and $Q(\mathbf{z}|\mathbf{Z})$ from $L(\mathbf{r},t|\mathbf{R})$, by (i) putting $\lambda = 0$, (ii) differentiating once with respect to t, and (iii) substituting $t-T$ for t.

Now, when $\lambda = 0$, (5.2) yields $K(\mathbf{r},t|\mathbf{R}) = H(t)\,M(\mathbf{r}|\mathbf{R})$:

$$M(\mathbf{r}|\mathbf{R}) = (4\pi)^{-1}\ln\left\{\frac{\sinh^2[\tfrac{1}{2}\pi(x-X)]+\sin^2[\tfrac{1}{2}\pi(y+Y)]}{\sinh^2[\tfrac{1}{2}\pi(x-X)]+\sin^2[\tfrac{1}{2}\pi(y-Y)]}\right\}; \tag{7.2}$$

while from (5.4)–(5.13), noting that $W_1 \equiv W_2 \equiv 0$ (i.e. both travelling waves W_1 and W_2 are vibrationally sustained by a non-zero source frequency λ), and appealing to (A 17), we arrive at

$$L(\mathbf{r},t|\mathbf{R}) = a_0^2 H(t)\left[\frac{N(\mathbf{r}_1|\mathbf{R})}{V(U+V)} - \frac{N(\mathbf{r}_2|\mathbf{R})}{V(U-V)} + \frac{2N(\mathbf{r}|\mathbf{R})}{U^2-V^2}\right], \tag{7.3}$$

provided $U \neq V$, with

$$N(\mathbf{r}|\mathbf{R}) = (4\pi)^{-1}\ln\left\{\frac{\sinh^2[\tfrac{1}{4}\pi(x-X)]+\sin^2[\tfrac{1}{4}\pi(y+Y)]}{\sinh^2[\tfrac{1}{4}\pi(x-X)]+\cos^2[\tfrac{1}{4}\pi(y-Y)]}\right\}, \tag{7.4}$$

and where $\mathbf{r}_1$ and $\mathbf{r}_2$ are defined by (4.12). In this case, $K(\mathbf{r},t|\mathbf{R}) + L(\mathbf{r},t|\mathbf{R})$ determines (via (7.2) and (7.4)) the ξ solution for the abruptly activated and constantly maintained filament source: $J(\mathbf{r},t) = \delta(\mathbf{r}-\mathbf{R})\,H(t)$.

Applying rules (ii) and (iii) above to (7.2),

$$P(\mathbf{z}|\mathbf{Z}) = \delta(t-T)\,M(\mathbf{r}|\mathbf{R}), \tag{7.5}$$

which is symmetric in y and Y (and $\mathbf{r}$, $\mathbf{R}$ as well). Likewise, from (7.3) and (7.4), and noting that the [ ] term in (7.3) vanishes at $t = 0$, we obtain

$$Q(\mathbf{z}|\mathbf{Z}) = H(t-T)\,Q_+(\mathbf{z}|\mathbf{Z}), \tag{7.6}$$

with

$$Q_+(\mathbf{z}|\mathbf{Z}) = G[x-X-(U-V)(t-T),y;\,Y]-G[x-X-(U+V)(t-T),y;\,Y], \tag{7.7}$$

where

$$G(x,y;\,Y) = \frac{(a_0^2/16V)\sinh(\frac{1}{2}\pi x)\cos(\frac{1}{2}\pi y)\cos(\frac{1}{2}\pi Y)}{\{\sinh^2(\frac{1}{4}\pi x)+\sin^2[\frac{1}{4}\pi(y+Y)]\}\{\sinh^2(\frac{1}{4}\pi x)+\cos^2[\frac{1}{4}\pi(y-Y)]\}}. \tag{7.8}$$

In using (7.3) to establish this result, we assumed $U \neq V$. Nevertheless, by starting independently from (5.30)–(5.35), it can be shown by similar arguments that (7.6)–(7.8) also hold whenever $U = V$. Whereupon, for every $U \geqslant 0$, (7.1) reduces, via (7.5) and (7.6), to

$$\xi(\mathbf{z}) = H(t)\left[\iint_{\Xi} J(\mathbf{R},t)\,M(\mathbf{r}|\mathbf{R})\,d\mathbf{R}+\iiint_{\mathscr{D}(t)} J(\mathbf{Z})\,Q_+(\mathbf{z}|\mathbf{Z})\,d\mathbf{Z}\right], \tag{7.9}$$

the general solution at time t within the infinite strip $\Xi:\{-\infty < x < \infty, 0 < y < 1\}$; the double integral ranges over Ξ, of which $d\mathbf{R}$ denotes an area element, while the triple integral ranges over the infinite hyperbar

$$\mathscr{D}(t):\{0 \leqslant T \leqslant t,\ -\infty < X < \infty, 0 < Y < 1\},$$

of which $d\mathbf{Z}$ is a volumetric element. The perturbation term $\iint_{\Xi}$ executes a spatially modified, in-phase time variation with the source current $J(\mathbf{r},t)$. It generalizes the kernel $K(\mathbf{r},t|\mathbf{R})$.

For a filament source current with an arbitrary time variation, viz.

$$J(\mathbf{z}) = \delta(\mathbf{r}-\mathbf{R})\,J(t)\,H(t), \tag{7.10}$$

$$\xi(\mathbf{z}) = H(t)\left[M(\mathbf{r}|\mathbf{R})\,J(t)+\int_0^t J(T)\,Q_+(\mathbf{r},t|\mathbf{R},T)\,dT\right]. \tag{7.11}$$

If the filament source acts impulsively at $t = T$ (> 0), i.e.

$$J(\mathbf{z}) = \delta(\mathbf{z}-\mathbf{Z}) \equiv \delta(x-X)\,\delta(y-Y)\,\delta(t-T), \tag{7.12}$$

$$\xi(\mathbf{z}) = \delta(t-T)\,M(\mathbf{r}|\mathbf{R})+H(t-T)\,Q_+(\mathbf{z}|\mathbf{Z}) = \begin{cases} Q_+(\mathbf{z}|\mathbf{Z}) & (t > T), \quad (7.13)\\ 0 & (t < T). \quad (7.14)\end{cases}$$

This solution is the basic Green's function for an arbitrary space–time distribution $J(\mathbf{r},t)$ (as opposed to the Green's function of (4.8) and (4.9), associated with the general oscillatory current source of (4.1)). The contribution

$$\delta(t-T)\,M(\mathbf{r}|\mathbf{R}) = P(\mathbf{z}|\mathbf{Z})$$

(cf. (7.5)) is completely in phase with the source, operating momentarily at $t = T$ with singular effect, and, like $K(\mathbf{r},t|\mathbf{R})$, possesses a logarithmic singularity about the generating filament through $\mathbf{r} = \mathbf{R}$ (see (7.2), also (5.2)). The contribution $H(t-T)\,Q_+(\mathbf{z}|\mathbf{Z}) = Q(\mathbf{z}|\mathbf{Z})$ (cf. (7.6)) operates only from the instant $t = T$ of 'switching on', a consequence of our part-time postulate. Like $L(\mathbf{r}, t|\mathbf{R})$ at zero λ frequency, $Q_+(\mathbf{z}|\mathbf{Z})$ is analytic for all $\mathbf{z}$, $\mathbf{Z}$ within the infinite hypertower $\mathscr{D}(\infty)$ (cf. (7.3), (7.4), (7.7), (7.8)).

8. Equivalence properties

Suppose the source current is *impulsive*, but arbitrarily distributed in space:

$$J(\mathbf{z}) = J(\mathbf{r})\,\delta(t-T) \quad (T > 0). \tag{8.1}$$

Whence, throughout $t < T$, $\xi(\mathbf{z}) \equiv 0$, by virtue of (7.9). Let us assume as from now that, unless otherwise stated, $t > T$. In this case,

$$\xi(\mathbf{z}) = \iint_{\Xi} J(\mathbf{R})\, Q_+(\mathbf{z}|\mathbf{Z})\, d\mathbf{R} \quad (\text{in } \Xi), \tag{8.2}$$

amounting to a generalization of the Green's function $Q_+(\mathbf{z}|\mathbf{Z})$ (for $t > T$). The latter is determined via (7.7), in terms of $G(x, y;\ Y)$ expressed by (7.8); note that

$$G(0, y;\ Y) \equiv 0, \quad G(x, y;\ Y) = -G(-x, y;\ Y). \tag{8.3}$$

Let the x, t (half) plane be confined to the domain above the horizontal initial axis $t = T$. Within this plane, the γ_1 and γ_2 (straight line) characteristics crossing any point $(x, t) = (X_0, T_0)$ are represented by (cf. § 5)

$$\left.\begin{aligned} \gamma_1(X_0, T_0):\quad x - X_0 = (U+V)(t-T_0),\\ \gamma_2(X_0, T_0):\quad x - X_0 = (U-V)(t-T_0). \end{aligned}\right\} \tag{8.4}$$

Along $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$, which extend only upwards from the initial point (X_0, T) into $t > T$, the Green's functions are, respectively,

$$\begin{aligned}(Q_+)_{\gamma_1(X_0, T)} &\equiv [Q_+(\mathbf{z}|\mathbf{Z})]_{\gamma_1(X_0, T)}\\ &= G[X_0 - X + 2V(t-T), y;\ Y] - G(X_0 - X, y;\ Y),\end{aligned} \tag{8.5}$$

$$\begin{aligned}(Q_+)_{\gamma_2(X_0, T)} &\equiv [Q_+(\mathbf{z}|\mathbf{Z})]_{\gamma_2(X_0, T)}\\ &= G(X_0 - X, y;\ Y) - G[X_0 - X - 2V(t-T), y;\ Y],\end{aligned} \tag{8.6}$$

valid for any flow speed $U \geqslant 0$. If $U = 0$, $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$ are symmetric about the vertical line $X = X_0$, along which

$$(Q_+)_{x=X_0} = G[X_0 - X + V(t-T), y;\ Y] - G[X_0 - X - V(t-T), y;\ Y], \tag{8.7}$$

which at any time $t = \tau + T$ becomes

$$(Q_+)_{x=X_0,\, t=\tau+T} = [(Q_+)_{\gamma_1(X_0, T)} + (Q_+)_{\gamma_2(X_0, T)}]_{t=\frac{1}{2}\tau+T}, \tag{8.8}$$

viz. the sum of (8.5) and (8.6) taken at $t = \frac{1}{2}\tau + T$. Applying this to (8.2), we arrive at the following.

THEOREM 1. (A time-difference law of equivalent propagation.) Under exposure to the general impulsive source of (8.1) in the presence of a stationary flow, $U = 0$, the ξ value for any given y within $(0, 1)$ at any instant $t = \tau + T$ $(\tau > 0)$ and along a vertical line $x = X_0$ starting from the initial point (X_0, T) equals the sum of the ξ values along the symmetric characteristics $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$ at the earlier instant $t = \frac{1}{2}\tau + T$.

In view of (8.3), (8.5) and (8.6) reduce, when $X_0 = X$, to

$$(Q_+)_{\gamma_1(X,T)} = G[2V(t-T), y; Y] = (Q_+)_{\gamma_2(X,T)} \quad (U \geqslant 0). \tag{8.9}$$

This result can be extended and generalized by the following theorem.

THEOREM 2. (An equivalence law of simultaneous propagation.) Suppose the spatial distribution of the impulsive source (8.1) is symmetric about some vertical line $x = X_0$, i.e.

$$J(X_0 - X, Y) = J(X_0 + X, Y). \tag{I}$$

Then along $\gamma_1(X_0, T)$ and $\gamma_2(X_0, T)$ diverging from the initial point on the given line, the ξ values at any instant are (simultaneously) equal for $U \geqslant 0$:

$$(\xi)_{\gamma_1(X_0,T)} = (\xi)_{\gamma_2(X_0,T)} = \iint_{\Xi} J(\mathbf{R})\, G[X - X_0 + 2V(t-T), y; Y]\, d\mathbf{R}, \tag{II}$$

with $G(x, y; Y)$ defined by (7.8). Furthermore, if $U = 0$, then along the two symmetric γ_1 and γ_2 characteristics crossing each other on the same line of symmetry $x = X_0$ at any other point, the ξ values are also equal at any instant. (Note that both (8.9) and (II) have obvious applications to (8.8) when $X_0 = X$ and theorem 1 holds respectively.)

Proof. Owing to the second identity of (8.3), (I) implies

$$\iint_{\Xi} J(\mathbf{R})\, G(X_0 - X, y; Y)\, d\mathbf{R} = 0, \tag{8.10}$$

$$\iint_{\Xi} J(\mathbf{R})\, G[X_0 - X + 2V(t-T), y; Y]\, d\mathbf{R}$$
$$= \iint_{\Xi} J(\mathbf{R})\, G[X - X_0 + 2V(t-T), y; Y]\, d\mathbf{R} \tag{8.11}$$
$$= -\iint_{\Xi} J(\mathbf{R})\, G[X_0 - X - 2V(t-T), y; Y]\, d\mathbf{R}. \tag{8.12}$$

Application to (8.2), (8.5) and (8.6) leads directly to (II). If $U = 0$, and (X_0, T_0) is any point (other than (X_0, T)) on $x = X_0$, then, from (7.7) and (8.4),

$$(Q_+)_{\gamma_1(X_0,T_0)} = G[X_0 - X + 2Vt - V(T_0 + T), y; Y] - G[X_0 - X - V(T_0 - T), y; Y],$$

$$(Q_+)_{\gamma_2(X_0,T_0)} = G[X_0 - X + V(T_0 - T), y; Y] - G[X_0 - X - 2Vt + V(T_0 + T), y; Y].$$

Following through with an argument like that based upon (8.11) and (8.12), we obtain

$$(\xi)_{\gamma_1(X_0,T_0)} = (\xi)_{\gamma_2(X_0,T_0)}, \tag{8.13}$$

completing the proof. (Note that the identity here does not involve the same integral convolution as (II).)

Suppose the γ_1 and γ_2 characteristics through an arbitrary point (x, t) are $\gamma_1(X_1, T_1)$, $\gamma_2(X_2, T_2)$. Let the γ_2 characteristic through (X_1, T_1) be $\gamma_2(X_3, T_3)$, (X_3, T_3) being the point such that $\gamma_1(X_3, T_3)$ passes exactly through (X_2, T_2). The two pairs of γ_1 and γ_2 characteristics, thus constructed, form what we shall call a *parallelogram of characteristics* with ordered vertices at (x,t), (X_1,T_1), (X_3,T_3), (X_2,T_2).

THEOREM 3. (Parallelogrammic equivalence.) The sum of the ξ values (associated with the general impulsive source current) at two opposite vertices of any parallelogram of characteristics equals the sum of the ξ values at the other two opposite vertices. The ξ value at any vertex lying on the initial axis $t = T$ is singular, being comparable with $\delta(t-T)$, but does not participate in the parallelogrammic equivalence above $t = T$, viz. the ξ value at the opposite vertex simply equals the sum of the ξ values at both adjacent vertices. (Note that, since U ($\geqslant 0$) and V (> 0) are finite velocities, the γ_1 and γ_2 characteristics are never horizontal. Hence a parallelogram of characteristics in $t > T$ has, at most, only one vertex along the initial axis $t = T$.)

Proof. We base our arguments on the typical parallelogram of characteristics described above. The points (x,t), (X_3,T_3) constitute a pair of opposite vertices, while the complementary pair comprises just (X_1,T_1) and (X_2,T_2). Now $\gamma_1(X_1,T_1)$ and $\gamma_2(X_2,T_2)$ intersect at (x,t); it therefore follows from (7.7) and (8.4) that

$$Q_+(\mathbf{z}|\mathbf{Z}) = G[X_2-X-(U-V)(T_2-T),y;\,Y] - G[X_1-X-(U+V)(T_1-T),y;\,Y].$$

Since (X_1,T_1) lies on $\gamma_2(X_3,T_3)$, while (X_2,T_2) lies on $\gamma_1(X_3,T_3)$,

$$G[X_1-X-(U-V)(T_1-T),y;\,Y]-G[X_3-X-(U-V)(T_3-T),y;\,Y] = 0,$$
$$G[X_3-X-(U+V)(T_3-T),y;\,Y]-G[X_2-X-(U+V)(T_2-T),y;\,Y] = 0.$$

Adding all three results, and turning once again to (7.7), we arrive at

$$Q_+(\mathbf{z}|\mathbf{Z})+Q_+(\mathbf{Z}_3|\mathbf{Z}) = Q_+(\mathbf{Z}_1|\mathbf{Z})+Q_+(\mathbf{Z}_2|\mathbf{Z}), \tag{8.14}$$

where $\mathbf{Z}_\nu = (X_\nu, y, T_\nu)$ with $\nu = 1, 2, 3$. Application to (8.2) yields

$$\xi(\mathbf{z})+\xi(\mathbf{Z}_3) = \xi(\mathbf{Z}_1)+\xi(\mathbf{Z}_2). \tag{8.15}$$

This proves the first part of the theorem, on the tacit assumption that

$$t > T \quad \text{and} \quad T_\nu > T \quad (\nu = 1,2,3).$$

Now, the complete ξ solution covering all values of t (> 0) is not given by (8.2), but is instead, via (7.9) and (8.1),

$$\xi(\mathbf{z}) = \delta(t-T)\iint_\Xi J(\mathbf{R})\,M(\mathbf{r}|\mathbf{R})\,d\mathbf{R}+H(t-T)\iint_\Xi J(\mathbf{R})\,Q_+(\mathbf{r},t|\mathbf{R},T)\,d\mathbf{R}. \tag{8.16}$$

Consequently, whenever the vertex (x,t), say, lies along the initial axis $t = T$, $\xi(\mathbf{z})$ acquires an infinitely dominant singular contribution of the order of $\delta(t-T)$. However, the formula (7.7) is valid at $t = T$ (although it no longer represents the Green's function), and reveals that

$$Q_+(\mathbf{z}|\mathbf{Z}) \equiv 0 \quad \text{when} \quad t = T, \tag{8.17}$$

in which case (8.14) reduces to

$$Q_+(\mathbf{Z}_3|\mathbf{Z}) = Q_+(\mathbf{Z}_1|\mathbf{Z})+Q_+(\mathbf{Z}_2|\mathbf{Z}). \tag{8.18}$$

But, provided $T_\nu > T$ ($\nu = 1, 2, 3$), (8.2) remains applicable at $\mathbf{z} = \mathbf{Z}_\nu$. So, if (x, t) lies along $t = T$, then

$$\xi(\mathbf{Z}_3) = \xi(\mathbf{Z}_1) + \xi(\mathbf{Z}_2). \tag{8.19}$$

The proof is now complete.

9. Asymptotic development

The general solution (7.9) can be asymptotically approximated for either large $|x|$, or large t, by spatially restricting the current distribution $J(\mathbf{z})$ to a time-independent, finite subdomain Σ of the infinite strip Ξ, and leaving it switched on over only a finite time interval $[0, \tau]$: $J(\mathbf{z}) \equiv 0$ whenever $t > \tau$, as well as outside Σ during $0 \leqslant t \leqslant \tau$. Throughout Σ and its boundary $\partial\Sigma$, we have $x_- \leqslant x \leqslant x_+$, x_+ (x_-) being the constant maximum (minimum) value of x over $\partial\Sigma$. To approximate (7.9), we employ (7.7) and (7.8). Thus, whenever $t > \tau$,

$$\begin{aligned}\xi \sim (a_0^2/2V)\cos(\tfrac{1}{2}\pi y)\Big\{&\operatorname{sgn}[x - x_- - (U-V)t]\int_0^\tau dT\iint_\Sigma J(X, Y, T)\cos(\tfrac{1}{2}\pi Y)\\ &\times\exp\{-\tfrac{1}{2}\pi|x - X - (U-V)(t-T)|\}\,dX\,dY\\ &-\operatorname{sgn}[x - x_+ - (U+V)t]\int_0^\tau dT\iint_\Sigma J(X, Y, T)\cos(\tfrac{1}{2}\pi Y)\\ &\times\exp\{-\tfrac{1}{2}\pi|x - X - (U+V)(t-T)|\}\,dX\,dY\Big\},\end{aligned} \tag{9.1}$$

valid for (see figures 4 and 5)

$$x \gg x_+ + (U+V)t \quad \text{(i.e. very far downstream of } \gamma_1(x_+, 0)), \tag{9.2}$$

$\gamma_\nu(X_0, T_0)$ ($\nu = 1, 2$) denoting the γ_ν characteristic through (X_0, T_0) with equation given by (8.4), as well as within any one of the following x, t zones.

Case $U > V$

$$x \ll x_- + (U-V)(t-\tau) \quad \text{(i.e. very far upstream of } \gamma_2(x_-, \tau)). \tag{9.3}$$

Case $U < V$

$$x \ll x_- + (U-V)t \quad \text{(i.e. very far upstream of } \gamma_2(x_-, 0)), \tag{9.4}$$

and

$$x_+ + (U-V)(t-\tau) \ll x \ll x_- + (U+V)(t-\tau). \tag{9.5}$$

At any finite instant, (9.2)–(9.4) describe appropriately large axial distances measured upstream and downstream from the domain Σ, in which case, (9.1) is an estimate for large $|x|$. In particular, we see that $\xi \to 0$ as $|x| \to \infty$. One implication is that no disturbances are being created at infinity, a compliance (once again) with the radiation principle. Furthermore, at any finite position, there eventually comes a time beyond which (9.1) holds (as a large-t approximation); this is implied by (9.3) for $U > V$, and by (9.5) for $U < V$. In this event, $\xi \to 0$ as $t \to \infty$, corresponding to an ultimate steady state of absolute 'silence', attained after an infinite period starting from the instant τ of switching-off.

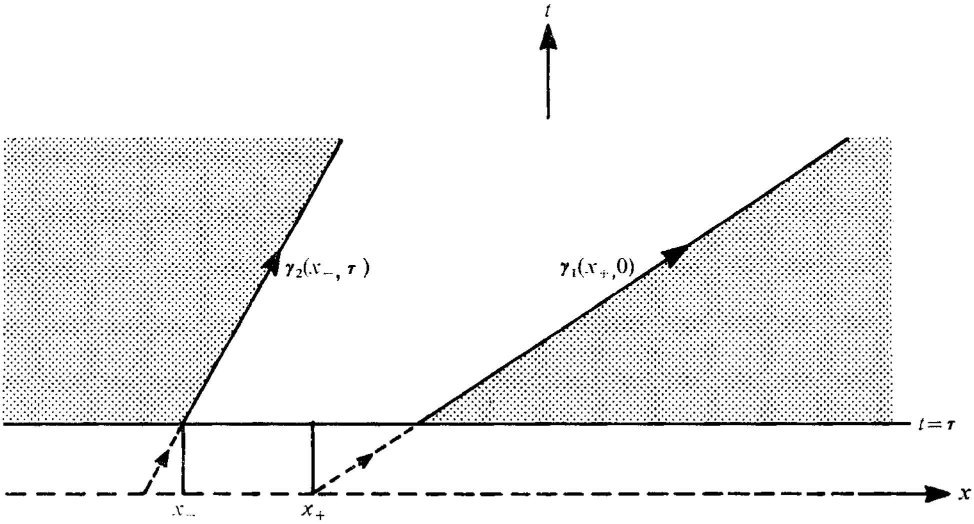

FIGURE 4. Case $U > V$. The zones of asymptotic validity for (9.1) occur within the shaded (partially infinite) regions, sufficiently far, in accordance with (9.2) and (9.3), from $\gamma_1(x_+, 0)$ and $\gamma_2(x_-, \tau)$, the two extreme characteristics about $t = \tau$, and emanating from an x, t section of the distribution $J(\mathbf{z})$.

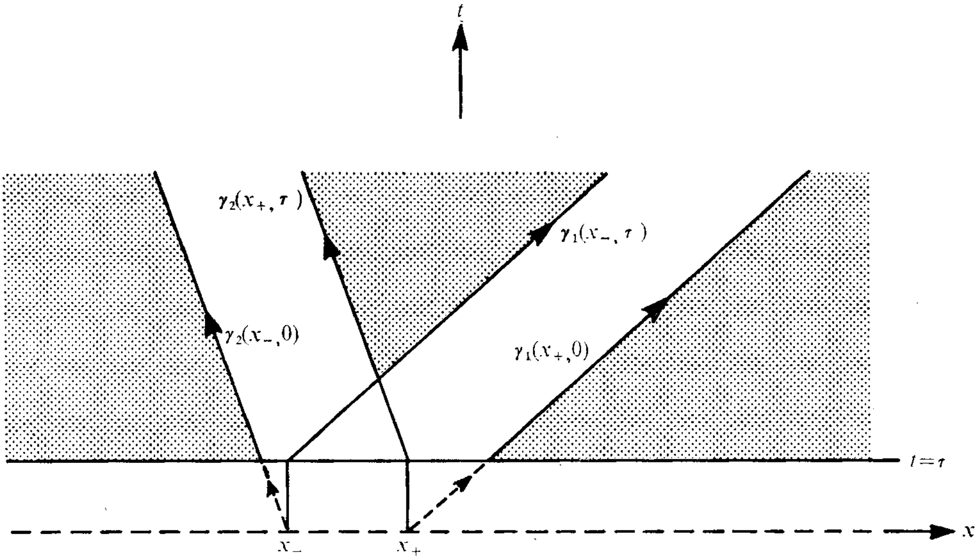

FIGURE 5. Case $U < V$. The shaded regions contain the zones defined by (9.2), (9.4) and (9.5). The approximation for large $|x|$ holds sufficiently far to the right and left of, respectively, $\gamma_1(x_+, 0)$ and $\gamma_2(x_-, 0)$, the relevant extreme characteristics. The same approximation (i.e. (9.1)), but for large t, holds sufficiently high above the two intersecting characteristics $\gamma_1(x_-, \tau)$ and $\gamma_2(x_+, \tau)$.

Appendix. The function $F(V;\mathbf{r}|\mathbf{R})$

The ω integral reduction of (4.6) leads to a formula (4.13) involving the function $F(V;\mathbf{r}|\mathbf{R})$. This is defined, throughout $0 \leqslant y \leqslant 1$ and $0 < Y < 1$, by

$$F(V;\mathbf{r}|\mathbf{R}) = \frac{1}{2\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sinh[\alpha(1-y)]\sinh[\alpha(1-Y)]}{\sinh(2\alpha)\,[(U+V)\,\alpha+\lambda]}\exp\{i\alpha(x-X)\}\,d\alpha, \quad \text{(A 1)}$$

in terms of a Cauchy principal value (PV), U, V and λ being real. The only singularities of the integrand, extended into the complex α plane, are a simple pole at

$$\alpha = \alpha_1 \equiv -\lambda/(U+V) \quad \text{along the given path} \quad (-\infty,\infty) \qquad \text{(A 2)}$$

(and hence the PV interpretation), together with an infinity of symmetric simple poles at

$$\alpha = \tfrac{1}{2}i\nu\pi, \quad -\tfrac{1}{2}i\nu\pi \quad (\nu = 1,2,3,\ldots,\infty) \text{ along the } \mathrm{Im}\,\alpha \text{ axis}, \qquad \text{(A 3)}$$

determined by the zeros of $\sinh(2\alpha)$. Though this factor vanishes at $\alpha = 0$, the integrand concerned is actually analytic near and at $\alpha = 0$, approaching zero as $\alpha \to 0$. Note that, whenever $y = 1$, $F(V;\mathbf{r}|\mathbf{R}) \equiv 0$; so we need to develop (A 1) only for $0 \leqslant y < 1$.

The above integral can be tackled by first continuing its path into the α plane. The usual appeal to Jordan's lemma (associated with an integration along a large semicircle expanding to infinity) is not directly feasible here, because the necessary uniform convergence principle fails along the $\mathrm{Im}\,\alpha$ axis owing to a blockage of this route by the distribution of singularities diverging both ways to $+i\infty$, $-i\infty$. For our purposes, then, we consider the related integral

$$I = \int_{\mathscr{L}} \frac{\sinh[\alpha(1-y)]\sinh[\alpha(1-Y)]}{\sinh(2\alpha)\,(\alpha-\alpha_1)}\exp\{i\alpha(x-X)\}\,d\alpha \quad (0 \leqslant y < 1,\ 0 < Y < 1), \qquad \text{(A 4)}$$

taken over the path $\mathscr{L}$, which does not pass through any of the poles given by (A 2) or (A 3). It can be proved that

$$|I| \leqslant \frac{1}{\min\limits_{\alpha\in\mathscr{L}}|\alpha-\alpha_1|}\int_{\mathscr{L}} \frac{\cosh[(1-y)\,|\mathrm{Re}\,\alpha|]\cosh[(1-Y)\,|\mathrm{Re}\,\alpha|]}{\sinh(2|\mathrm{Re}\,\alpha|)} \times \exp\{-(x-X)\,\mathrm{Im}\,\alpha\}\,|d\alpha|, \qquad \text{(A 5)}$$

$$< \frac{1}{2\min\limits_{\alpha\in\mathscr{L}}|\alpha-\alpha_1|}\int_{\mathscr{L}} \coth|\mathrm{Re}\,\alpha|\exp\{-(x-X)\,\mathrm{Im}\,\alpha\}\,|d\alpha|, \qquad \text{(A 6)}$$

with $\min\limits_{\alpha\in\mathscr{L}}|\alpha-\alpha_1|$ denoting the minimum (a perpendicular) distance of α_1 from $\mathscr{L}$. Consider the vertical rectangular boundary of height $2N$ and width $2N^{\frac{1}{2}}$, formed from the straight paths $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_8$ (see figure 6). $\mathscr{L}_1$ is joined to $\mathscr{L}_2$, and $\mathscr{L}_3$ to $\mathscr{L}_4$, via semicircular indentations, each of radius $\epsilon = \sinh^{-1}(\frac{1}{2}\exp\{-N^{\frac{1}{2}}\})$, centred respectively at iN, $-iN$. Suppose

$$N^{\frac{1}{2}} > \max\{|\alpha_1|, \ln(2\sinh\tfrac{1}{4}\pi)^{-1}\}, \qquad \text{(A 7)}$$

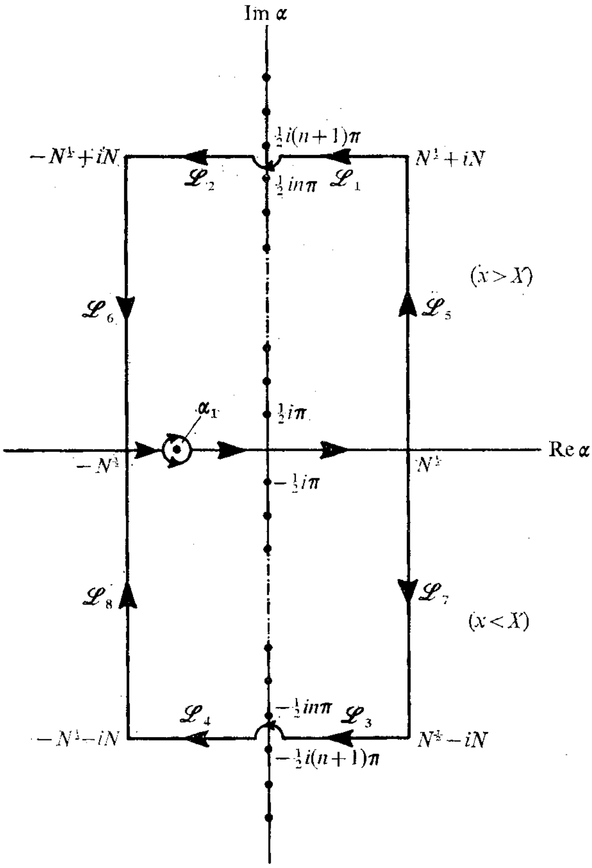

FIGURE 6. The vertical, indented, rectangular boundary, whose upper/lower half in $\operatorname{Im}\alpha \gtrless 0$, joined to the indented real base $(-N^{\frac{1}{2}}, N^{\frac{1}{2}})$, provides a closed contour for determining $F(V;\mathbf{r}|\mathbf{R})$ when $x \gtrless X$. There is no loss of generality in assuming $\lambda > 0$, in which case the pole α_1 is on the negative half of the Re α axis. (If $\lambda < 0$, α_1 is on the positive half of this axis, but leads to the same result.)

and let $N = \frac{1}{2}(n+\frac{1}{2})\pi$, n being a positive integer. Now, (A 7) implies, in particular, $\epsilon < \frac{1}{4}\pi$; i.e. each of the semicircular detours and its centre never coincide with any of the imaginary poles listed in (A 3). In fact, the closed indented rectangular boundary completely surrounds $2n$ of these imaginary poles, viz. $\frac{1}{2}i\nu\pi$, $-\frac{1}{2}i\nu\pi$ $(\nu = 1, \ldots, n)$, as well as the real pole α_1 (depicted as being negative, assuming $\lambda > 0$). Furthermore, as $N \to \infty$ (through integral values of n), this closed boundary expands to infinity while both its indentations contract towards their centres. In this limiting process, the advancing path hurdles intermittently across each remaining imaginary pole without ever making contact. All poles are thus eventually enclosed. Now both the integrands arising in (A 5) and (A 6) are largely dominated by the exponent $\exp\{-(x-X)\operatorname{Im}\alpha\}$. Hence, if any of the segments $\mathscr{L}_\nu$ $(\nu = 1, \ldots, 8)$ were to replace $\mathscr{L}$, then for convergence of I when $n \to \infty$, we require that

$$x \gtrless X \quad \text{whenever} \quad \operatorname{Im}\alpha \gtrless 0 \text{ on } \mathscr{L}. \tag{A 8}$$

For example, then, if $\mathscr{L} = \mathscr{L}_\nu$ ($\nu = 1, 2, 3, 4$), a horizontal segment, (A 6) yields

$$|I| < \frac{\exp\{-|x-X|N\}}{2N}\int_\epsilon^{N^{\frac{1}{2}}} \coth\zeta\, d\zeta = \frac{\exp\{-|x-X|N\}}{2N}\ln\left(\frac{\sinh N^{\frac{1}{2}}}{\sinh\epsilon}\right)$$
$$< \frac{\exp\{-|x-X|N\}}{2N}\ln(\exp\{2N^{\frac{1}{2}}\}) = \frac{\exp\{-|x-X|N\}}{N^{\frac{1}{2}}} \to 0 \quad \text{as} \quad N\to\infty. \tag{A 9}$$

(Note that the integrand $\coth\zeta$ is singular at $\zeta = 0$, but not at $\zeta = \epsilon$; hence the reason for having the indentations about $\alpha = iN, -iN$.) On the other hand, if $\mathscr{L} = \mathscr{L}_\nu$ ($\nu = 5, 6, 7, 8$), a vertical segment, then using (A 5) instead, we get

$$|I| < \frac{\cosh[(1-y)N^{\frac{1}{2}}]\cosh[(1-Y)N^{\frac{1}{2}}]}{\sinh(2N^{\frac{1}{2}})(N^{\frac{1}{2}}-|\alpha_1|)}\int_0^N \exp\{-|x-X|\zeta\}d\zeta$$
$$< \frac{2\exp\{-(y+Y)N^{\frac{1}{2}}\}}{|x-X|(N^{\frac{1}{2}}-|\alpha_1|)(1-\exp\{-4N^{\frac{1}{2}}\})} \to 0 \quad \text{as} \quad N\to\infty. \tag{A 10}$$

Now, the integrand occurring in (A 4) is analytic at $\alpha = \pm iN$ and on the semi-circular indentation: $|\alpha \mp iN| = \epsilon$, along which its magnitude is therefore bounded above by $M_\pm$ (>0), say. Consequently, if $\mathscr{L}$ is replaced by this indentation path,

$$|I| < M_\pm \pi\epsilon = M_\pm\pi\sinh^{-1}(\tfrac{1}{2}\exp\{-N^{\frac{1}{2}}\}) \to 0 \quad \text{as} \quad N\to\infty. \tag{A 11}$$

To determine $F(V;\mathbf{r}|\mathbf{R})$ from (A 1), we select, corresponding to $x \gtrless X$ and in accordance with (A 8), a closed contour comprising the upper/lower (i.e. $\operatorname{Im}\alpha \gtrless 0$) half of the indented rectangular path of figure 6, completed by the real base $(-N^{\frac{1}{2}}, N^{\frac{1}{2}})$, which is indented about the pole at $\alpha = \alpha_1$ by a semicircle protruding slightly into $\operatorname{Im}\alpha \gtrless 0$. This closed contour, thus constructed, is positively/negatively directed, and circumscribes the n imaginary poles at

$$\alpha = \pm\tfrac{1}{2}i\nu\pi \quad (\nu = 1, \ldots, n),$$

but excludes the real pole α_1. We apply the residue theorem; let $N\to\infty$, taking into account the results (A 9)–(A 11); and, consistent with a PV, squeeze in, to the point of vanishing, each indentation about α_1. Whereupon, we arrive at

$$\text{if} \quad x > X: \qquad \text{PV}\int_{-\infty}^{\infty}(\;) = \pi i \operatorname*{residue}_{\alpha_1}(\;) + 2\pi i\sum_{\nu=1}^{\infty}\operatorname*{residue}_{\frac{1}{2}i\nu\pi}(\;), \tag{A 12}$$

$$\text{but if} \quad x < X: \quad \text{PV}\int_{-\infty}^{\infty}(\;) = -\pi i \operatorname*{residue}_{\alpha_1}(\;) - 2\pi i\sum_{\nu=1}^{\infty}\operatorname*{residue}_{-\frac{1}{2}i\nu\pi}(\;), \tag{A 13}$$

where ( ) represents the integrand of (A 4). Application to (A 1) leads to

$$F(V;\mathbf{r}|\mathbf{R}) = i\,\operatorname{sgn}(x-X)\frac{\sinh[\alpha_1(1-y)]\sinh[\alpha_1(1-Y)]\exp\{i\alpha_1(x-X)\}}{2(U+V)\sinh(2\alpha_1)}$$
$$-iH(x-X)\sum_{n=1}^{\infty}\frac{(-1)^n\sin[\frac{1}{2}n\pi(1-y)]\sin[\frac{1}{2}n\pi(1-Y)]\exp\{-\frac{1}{2}n\pi|x-X|\}}{2\lambda+in\pi(U+V)}$$
$$+iH(X-x)\sum_{n=1}^{\infty}\frac{(-1)^n\sin[\frac{1}{2}n\pi(1-y)]\sin[\frac{1}{2}n\pi(1-Y)]\exp\{-\frac{1}{2}n\pi|x-X|\}}{2\lambda-in\pi(U+V)}. \tag{A 14}$$

The occurrence of the signum function sgn $(x-X)$ and both Heaviside functions apparently suggests a singular behaviour, viz. a discontinuity about $x = X$, following the sign disparity of the forms (A 12) and (A 13). From these, we observe that

$$\left[\mathrm{PV}\int_{-\infty}^{\infty}(\;)\right]_{x=X+0_+}-\left[\mathrm{PV}\int_{-\infty}^{\infty}(\;)\right]_{x=X+0_-}$$
$$=2\pi i\left[\underset{\alpha_1}{\mathrm{residue}}(\;)+\lim_{n\to\infty}\sum_{\nu=1}^{n}\underset{\frac{1}{2}i\nu\pi}{\mathrm{residue}}(\;)+\underset{-\frac{1}{2}i\nu\pi}{\mathrm{residue}}(\;)\right]_{x=X}$$
$$=\lim_{N\to\infty}\oint(\;)_{x=X}\quad\text{(via residue theory)},$$

provided $\oint(\;)_{x=X}$, which is performed over the closed indented rectangular boundary of figure 6 (but described, instead, in a fully positive sense), exists in the limit as $N\to\infty$. When $x = X$, the results of (A 9) as well as (A 11) remain valid. However, instead of the second inequality in (A 10), we have

$$|I|_{x=X}<\frac{2N^{\frac{1}{2}}\exp\{-(y+Y)N^{\frac{1}{2}}\}}{(1-|\alpha_1|/N^{\frac{1}{2}})(1-\exp\{-4N^{\frac{1}{2}}\})},$$

which therefore still $\to 0$ as $N\to\infty$. Consequently,

$$\lim_{N\to\infty}\oint(\;)_{x=X}\equiv 0,$$

so that, contrary to expectation, the integral $\mathrm{PV}\int_{-\infty}^{\infty}(\;)$ is continuous across $x = X$. Hence,

$$F(V;\mathbf{r}|\mathbf{R})|_{x=X+0_+}\equiv F(V;\mathbf{r}|\mathbf{R})|_{x=X+0_-}. \tag{A 15}$$

So $F(V;\mathbf{r}|\mathbf{R})$ is definitely continuous throughout $-\infty < x < \infty$. Both (A 14) and (A 15) are valid over $0 \leqslant y \leqslant 1$ and $0 < Y < 1$. (The expression (4.13) also includes the function $F(-V;\mathbf{r}|\mathbf{R})$. Regarding its integral representation, the integrand has a single real pole at $\lambda/(V-U)$ which, if $\lambda > 0$, lies along the positive/negative half of the $\mathrm{Re}\,\alpha$ axis (the path of integration), depending on $U \lessgtr V$. The final form for $F(-V;\mathbf{r}|\mathbf{R})$, however, is obtainable from (A 14) by merely substituting $-V$ for V, irrespective of $U \lessgtr V$.)

The above results fail when $V = -U$. In this case, (A 1) reveals that the real pole $\alpha = \alpha_1$ is missing, and so the PV interpretation is no longer needed. Otherwise, a similar contour integration technique is applicable and eventually leads to

$$F(-U;\mathbf{r}|\mathbf{R})=\frac{\mathrm{sgn}\,(x-X)}{2i\lambda}\sum_{n=1}^{\infty}(-1)^n\sin[\tfrac{1}{2}n\pi(1-y)]\sin[\tfrac{1}{2}n\pi(1-Y)]\times\exp\{-\tfrac{1}{2}n\pi|x-X|\}. \tag{A 16}$$

From a series formula of Gradshteyn & Ryzhik (1965, 1.462), it can be shown that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(2nx)\sin(2ny)\exp\{-2n|t|\}=\tfrac{1}{4}\ln\left[\frac{\sinh^2 t+\cos^2(x+y)}{\sinh^2 t+\cos^2(x-y)}\right]. \tag{A 17}$$

Taking its t derivative, then applying this to (A 16), we obtain the closed form

$$F(-U;\mathbf{r}|\mathbf{R}) = \frac{i(16\lambda)^{-1}\sinh[\frac{1}{2}\pi(x-X)]\cos(\frac{1}{2}\pi y)\cos(\frac{1}{2}\pi Y)}{\{\sinh^2[\frac{1}{4}\pi(x-X)]+\sin^2[\frac{1}{4}\pi(y+Y)]\}\{\sinh^2[\frac{1}{4}\pi(x-X)]+\cos^2[\frac{1}{4}\pi(y-Y)]\}}. \tag{A 18}$$

This demonstrates categorically that, when $0 < Y < 1$ and $|X| < \infty$, $F(-U;\mathbf{r}|\mathbf{R})$ is an analytic function of x and y throughout the (real) infinite strip

$$0 \leqslant y \leqslant 1: |x| < \infty.$$

We next consider (4.4). This is reducible via (4.11) to

$$K(\mathbf{r},t|\mathbf{R}) = H(t)\frac{\exp\{i\lambda t\}}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh[\alpha(1-y)]\sinh(\alpha Y)}{\alpha\sinh\alpha}\exp\{i\alpha(x-X)\}\,d\alpha. \tag{A 19}$$

In the α plane, the integrand is actually analytic at $\alpha = 0$ (in spite of a double zero at $\alpha = 0$ possessed by its denominator), but has an infinity of purely imaginary simple poles at $\alpha = i\nu\pi$, $-i\nu\pi$ ($\nu = 1,2,\ldots,\infty$). Provided the condition $Y \leqslant y$ but $\mathbf{r} \neq \mathbf{R}$ holds, the infinite integral involved is convergent and can again be tackled by a contour integration to yield

$$K(\mathbf{r},t|\mathbf{R}) = H(t)\pi^{-1}\exp\{i\lambda t\}\sum_{n=1}^{\infty} n^{-1}\sin(n\pi y)\sin(n\pi Y)\exp\{-n\pi|x-X|\}. \tag{A 20}$$

This series can be summed by employing (A 17). Thus,

$$K(\mathbf{r},t|\mathbf{R}) = H(t)\,(4\pi)^{-1}\exp\{i\lambda t\}\ln\left\{\frac{\sinh^2[\frac{1}{2}\pi(x-X)]+\sin^2[\frac{1}{2}\pi(y+Y)]}{\sinh^2[\frac{1}{2}\pi(x-X)]+\sin^2[\frac{1}{2}\pi(y-Y)]}\right\}, \tag{A 21}$$

which, for $0 < Y < 1$ and $|X| < \infty$, is influenced by a singular effect of $O(\ln|\mathbf{r}-\mathbf{R}|^2)$ near the point $\mathbf{r} = \mathbf{R}$, but stays analytic elsewhere inside

$$O \leqslant y \leqslant 1: |x| < \infty.$$

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